

Worcester County Mathematics League

Varsity Meet 2 - December 3, 2025

COACHES' COPY ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 2- December 3, 2025
Answer Key



Round 1 - Fractions, Decimals, and Percents

1. 64
2. 1232
3. $\frac{7770}{4433}$

Round 2 - Algebra I

1. 14
2. 3
3. 0.75 or $\frac{3}{4}$

Round 3 - Parallel Lines and Polygons

1. 49
2. (10, 144), exact order
3. $8\sqrt{2} - 8$ or $8(\sqrt{2} - 1)$

Round 4 - Sequences and Series

1. -6
2. 6201
3. 57

Round 5 - Matrices and Systems of Equations

1. 2
2. $\left(\frac{6}{7}, 3, 6\right)$, exact order
3. 36

Team Round

1. $\frac{77}{144}$
2. 2
3. $\frac{47}{61}$
4. 215
5. 1.6
6. 20
7. 58
8. $\frac{3}{4}$
9. $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

Worcester County Mathematics League
Varsity Meet 2- December 3, 2025
Round 1 - Fractions, Decimals, and Percents



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. In the Spring of 2019, 240 students from Worcester County high schools studied calculus. Of those students, 40% took the Calculus AB AP exam (which is scored 1 through 5). Of the students taking that exam, $\frac{1}{3}$ received a score below 3. How many of the students received a score of 3 or above?

2. N students were enrolled at Cartesian College at the beginning of the first semester. Enrollment increased by 48 students by the end of the first semester and decreased by 5% from the end of the first semester to the end of the second semester, at which time 1216 students were enrolled. Find N .

3. If $a = 2.\overline{82}$ and $b = 1.\overline{613}$, express $a \div b$ as a fraction (not a mixed number) in simplest terms.

ANSWERS

(1 pt) 1. _____ students

(2 pts) 2. $N =$ _____

(3 pts) 3. $\frac{a}{b} =$ _____

Athol/QSC, Tantasqua, Quaboag

Worcester County Mathematics League
Varsity Meet 2- December 3, 2025
Round 2 - Algebra I



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Laura has a 90 pound calf that she is raising for a 4H project. If the calf gains 65 pounds a month, in how many months will the calf weigh 1000 pounds?

2. Let $4x + y = 3$, $2x - 4y = -4$ and $3x + ky = 5$. Find k .

3. Jack had a jar of tea and Padma had a pitcher of tea. They took turns pouring a fraction of the tea from their container into the other person's container. Jack began by pouring half his jar into the pitcher, after which Padma poured a third of her pitcher into the jar. Then Jack poured a quarter of his jar into the pitcher. After Padma poured a fifth of her pitcher into the jar, there was $0.65l$ in the jar and $0.60l$ in the pitcher. If no tea was spilled in the pouring, how much tea, in l (liters), was in Jack's jar before any tea was poured?

ANSWERS

(1 pt) 1. _____ months

(2 pts) 2. $k =$ _____

(3 pts) 3. _____ l .

Southbridge, Mass. Acad., QSC

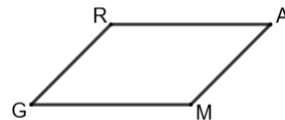
Worcester County Mathematics League
Varsity Meet 2- December 3, 2025
Round 3 - Parallel Lines and Polygons



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. $GRAM$ is a parallelogram, shown at right; $m\angle G = x + 10^\circ$ and $m\angle R = 3x + 14^\circ$.
Find $m\angle A$ in $^\circ$.



2. A regular polygon with n sides has interior angles with measure x° . A regular polygon with $n + 2$ sides has interior angles with measure $(x + 6)^\circ$. Find the ordered pair (n, x) .
3. A circle is inscribed in a regular octagon. The ratio of the area of the octagon to the area of the circle can be expressed as $\frac{x}{\pi}$. Find x .

ANSWERS

(1 pt) 1. $m\angle A = \underline{\hspace{2cm}}^\circ$

(2 pts) 2. $(n, x) = \left(\underline{\hspace{2cm}} \right)$

(3 pts) 3. $x = \underline{\hspace{2cm}}$

Worcester County Mathematics League
Varsity Meet 2- December 3, 2025
Round 4 - Sequences and Series



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. The numbers 2, n , and 18, in that order, form a geometric sequence. Find the smallest possible value of n .

2. Find S , the sum of the arithmetic sequence shown below:

$$S = 13 + 17 + 21 + \cdots + 217 + 221$$

3. Let $\sqrt{1 + 2 + 3 + \cdots + n}$ be a positive integer (as is n). Find the sum of the two smallest possible values for n , where $n > 1$.

ANSWERS

(1 pt) 1. _____

(2 pts) 2. $S =$ _____

(3 pts) 3. _____

Worc. Acad./QSC, St. John's, Hopkinton/QSC

Worcester County Mathematics League
Varsity Meet 2- December 3, 2025
Round 5 - Matrices and Systems of Equations



All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Solve for k :

$$\begin{vmatrix} -3 & k \\ 7 & -8 \end{vmatrix} = 10$$

2. Given the system of equations below, find the ordered triple (x, y, z) :

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{3}{2} \\ \frac{1}{x} + \frac{1}{z} &= \frac{4}{3} \\ \frac{1}{y} + \frac{1}{z} &= \frac{1}{2} \end{aligned}$$

3. Let

$$\begin{bmatrix} 2 & b \\ a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for some x and y , where a and b are integers and at least one of x and y does not equal zero. Find the sum of $a + b$ values over all possible ordered pairs (a, b) .

ANSWERS

(1 pt) 1. $k =$ _____

(2 pts) 2. $(x, y, z) = \left(\text{_____} \right)$

(3 pts) 3. _____

Doherty, Doherty, Doherty/QSC



All answers must be in simplest exact form in the answer section.

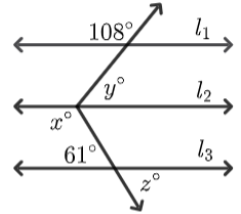
NO CALCULATORS ALLOWED

1. Let $x = \left(\frac{3}{4} + \frac{1}{6}\right) \left(\frac{3}{4} - \frac{1}{6}\right)$. Express x as a fraction in simplest terms.

2. Let $ab = a - b$ where a and b are two non-zero real numbers. Find

$$\frac{a}{b} + \frac{b}{a} - ab$$

3. Given three parallel lines in the figure at right, that is $l_1 \parallel l_2 \parallel l_3$, and the angle measures marked in the figure, find the value of $\frac{x-y}{z}$.



4. Find integer n where:

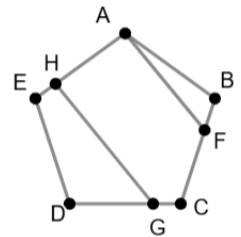
$$\frac{1 + 2 + 3 + \cdots + n}{3n} = 36.$$

5. Express x as a decimal if:

$$x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + 1}}}$$

6. How many ml of water must be added to $60ml$ of a 12% alcohol solution to produce a 9% solution?

7. $ABCDE$, shown at right, is a regular pentagon. F lies on $\overline{BC}$, G lies on $\overline{CD}$, and H lies on $\overline{EA}$. Also, $\overline{AF} \parallel \overline{HG}$ and $m\angle BAF = 22^\circ$. Find $m\angle HGD$ in $^\circ$.



8. Find S , shown below. Express your answer as a fraction in simplest form.

$$S = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

9. Find matrix A where

$$A \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} + \left(\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} \right)^T = \begin{bmatrix} 1 & 8 \\ 2 & 2 \end{bmatrix}$$

Worcester County Mathematics League
Varsity Meet 2- December 3, 2025
Team Round Answer Sheet



ANSWERS

1. $x =$ _____

2. _____

3. $\frac{x-y}{z} =$ _____

4. _____

5. $x =$ _____

6. _____ ml

7. $m\angle HGD =$ _____ $^{\circ}$

8. $S =$ _____

9. $A = \begin{bmatrix} & \\ & \end{bmatrix}$

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Round 1 - Fractions, Decimals, and Percents

1. In the Spring of 2019, 240 students from Worcester County high schools studied calculus. Of those students, 40% took the Calculus AB AP exam (which is scored 1 through 5). Of the students taking that exam, $\frac{1}{3}$ received a score below 3. How many of the students received a score of 3 or above?

Solution: Note that $40\% = \frac{40}{100} = 0.4$. Then 40% of 240 students is $0.4 \cdot 240 = 4 \cdot 24 = 96$ students. Note that if one third of those students score below 3, then two thirds of the students receive a score of 3 or above. Then $\frac{2}{3} \cdot 96 = 2 \cdot \frac{96}{3} = 2 \cdot 32 = \boxed{64}$.

Alternatively, one third of 96 students is $\frac{1}{3} \cdot 96 = \frac{96}{3} = 32$ students. Then the number of students that scored 3 or above is equal to the number of students who took the exam minus the number of students who scored less than 3, or $96 - 32 = 64$.

2. N students were enrolled at Cartesian College at the beginning of the first semester. Enrollment increased by 48 students by the end of the first semester and decreased by 5% from the end of the first semester to the end of the second semester, at which time 1216 students were enrolled. Find N .

Solution: Let M be the number of students enrolled at the end of the first semester. Start at the end of the second semester with an enrollment of 1216 students, find M , then find N .

Note that decreasing by 5% means that the number of students at the end of the second semester is 95% of the number at the end of the first semester. Then $95\% = \frac{95}{100} = \frac{5 \cdot 19}{5 \cdot 20} = \frac{19}{20}$, and $1216 = \frac{19}{20} \cdot M$. Solving, $M = \frac{20}{19} \cdot 1216 = 20 \cdot \frac{1216}{19} = 20 \cdot \frac{64 \cdot 19}{19} = 20 \cdot 64 = 1280$. Finally, note that $M = N + 48$, or $N = M - 48 = 1280 - 48 = \boxed{1232}$.

3. If $a = 2.\overline{82}$ and $b = 1.\overline{613}$, express $a \div b$ as a fraction (not a mixed number) in simplest terms.

Solution: Convert a from a decimal to a fraction by calculating $100a - a$ and noting that the repeating decimals cancel:

$$\begin{array}{r} 100a = 282.\overline{82} \\ -a = -2.\overline{82} \\ \hline 99a = 280 \end{array}$$

Therefore $a = \frac{280}{99}$. Next, convert b to a fraction by calculating $1000b - b$:

$$\begin{array}{r} 1000b = 1613.\overline{613} \\ -b = -1.\overline{613} \\ \hline 999b = 1612 \end{array}$$

Therefore $b = \frac{1612}{999}$. Recall that dividing by a fraction is equivalent to multiplying by the reciprocal of the fraction. The reciprocal of b is $\frac{999}{1612}$, so:

$$\begin{aligned} a \div b &= \frac{280}{99} \cdot \frac{999}{1612} \\ &= \frac{280 \cdot 999}{99 \cdot 1612} \\ &= \frac{280}{1612} \cdot \frac{999}{99} \\ &= \frac{4 \cdot 70}{4 \cdot 403} \cdot \frac{9 \cdot 111}{9 \cdot 11} \\ &= \frac{70}{403} \cdot \frac{111}{11} \\ &= \frac{70 \cdot 111}{403 \cdot 11} = \boxed{\frac{7770}{4433}} \end{aligned}$$

Round 2 - Algebra I

1. Laura has a 90 pound calf that she is raising for a 4H project. If the calf gains 65 pounds a month, in how many months will the calf weigh 1000 pounds?

Solution: Let r be the rate at which the calf gains weight: $r = 65$ pounds per month. Let m be the number of months until the calf weighs 1000 pounds. The amount of pounds that the calf needs to gain to reach 1000 is $1000 - 90 = 910$ pounds. Recall the rate equation: rate times time equals amount, or

$$r \cdot m = 910.$$

Divide both sides of the rate equation by r , substitute the value $r = 65$, and solve for m :

$$m = 910 \div r = \frac{910}{65} = \frac{13 \cdot 70}{13 \cdot 5} = \frac{70}{5} = \boxed{14} \text{ months.}$$

2. Let $4x + y = 3$, $2x - 4y = -4$ and $3x + ky = 5$. Find k .

Solution: Note that k can be found by solving the first two equations for x and y , substituting those values in the third equation, and then solving for k . Then subtract 2 times the second equation from the first equation to eliminate x and solve for y :

$$\begin{array}{r} 4x + y = 3 \\ -(4x - 8y = -8) \\ \hline 9y = 11 \end{array}$$

Therefore $y = \frac{11}{9}$. Substitute this value in the first equation: $4x + \frac{11}{9} = 3$. Then $4x = 3 - \frac{11}{9} = \frac{27-11}{9} = \frac{16}{9}$ and $x = \frac{1}{4} \cdot \frac{16}{9} = \frac{4}{9}$.

Next, substitute these values for x and y into the third equation and $3 \cdot \frac{4}{9} + k \cdot \frac{11}{9} = 5$, or $k \cdot \frac{11}{9} = 5 - 3 \cdot \frac{4}{9} = 5 - \frac{4}{3} = \frac{15-4}{3} = \frac{11}{3}$. Then $k = \frac{9}{11} \cdot \frac{11}{3} = \frac{9 \cdot 11}{11 \cdot 3} = \frac{3 \cdot 33}{33} = \boxed{3}$.

Alternative Solution: Divide the second equation by 2: $x - 2y = -2$. Subtract this equation from the first original equation:

$$\begin{array}{r} 4x + y = 3 \\ -(x - 2y = -2) \\ \hline 3x + 3y = 5 \end{array}$$

Compare this equation to the third equation, and $k = 3$.

3. Jack had a jar of tea and Padma had a pitcher of tea. They took turns pouring a fraction of the tea from their container into the other person's container. Jack began by pouring half his jar into the pitcher, after which Padma poured a third of her pitcher into the jar. Then Jack poured a quarter of his jar into the pitcher. After Padma poured a fifth of her pitcher into the jar, there was $0.65l$ in the jar and $0.60l$ in the pitcher. If no tea was spilled in the pouring, how much tea, in l (liters), was in Jack's jar before any tea was poured?

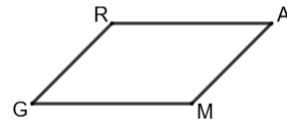
Solution: There were four pourings. Let j_n be the amount of tea in Jack's jar, measured in liters, after the n^{th} pouring. Let p_n be number of liters of tea in Padma's pitcher after the n^{th} pouring. Work backwards to find j_0 , starting from $j_4 = 0.65l$ and $p_4 = 0.6l$, while noting that $j_k + p_k = 0.65 + 0.6 = 1.25l$ for any pouring. That is, the total amount of tea doesn't change.

Because Padma poured a fifth of her pitcher (after the fourth pouring), p_4 is four fifths of p_3 : $\frac{4}{5}p_3 = p_4 = .6$, or $p_3 = \frac{5}{4}(.6) = \frac{5}{4}(\frac{3}{5}) = \frac{3}{4} = 0.75l$. That is, $p_3 = 0.75l$ and $j_3 = 1.25 - 0.75 = 0.5l$. Likewise, j_3 is three fourths of j_2 (after the third pouring): $\frac{3}{4}j_2 = .5$, or $j_2 = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3}l$.

Then $p_2 = 1.25 - \frac{2}{3} = \frac{5}{4} - \frac{2}{3} = \frac{15}{12} - \frac{8}{12} = \frac{7}{12}l$. Proceeding, p_2 is two thirds of p_1 and $\frac{2}{3}p_1 = \frac{7}{12}$, or $p_1 = \frac{3}{2} \cdot \frac{7}{12} = \frac{3 \cdot 7}{2 \cdot 3 \cdot 4} = \frac{7}{8}l$. Then $j_1 = \frac{5}{4} - \frac{7}{8} = \frac{10}{8} - \frac{7}{8} = \frac{3}{8}l$. Finally, $\frac{1}{2}j_0 = j_1 = \frac{3}{8}l$, and $j_0 = 2 \cdot \frac{3}{8} = \frac{3}{4} = \boxed{0.75}l$.

Round 3 - Parallel Lines and Polygons

1. $GRAM$ is a parallelogram, shown at right; $m\angle G = x + 10^\circ$ and $m\angle R = 3x + 14^\circ$. Find $m\angle A$ in $^\circ$.



Solution: Recall that a parallelogram is a quadrilateral with two pairs of parallel sides. Therefore $\overline{RA} \parallel \overline{GM}$. Note that $\angle G$ and $\angle R$ are same side interior angles and are therefore supplementary. That is, their measures sum to 180° : $m\angle G + m\angle R = (x + 10) + (3x + 14) = x + 3x + 10 + 14 = 4x + 24 = 180^\circ$. Solve for x : $4x = 180 - 24 = 156$ and $x = \frac{156}{4} = 39$. Recall that opposite angles in a parallelogram are congruent, that is they have the same measure. Then $m\angle A = m\angle G = x + 10 = 39 + 10 = \boxed{49}^\circ$.

2. A regular polygon with n sides has interior angles with measure x° . A regular polygon with $n + 2$ sides has interior angles with measure $(x + 6)^\circ$. Find the ordered pair (n, x) .

Solution: Recall that a regular polygon with n sides has interior angles that measure $\frac{n-2}{n} \cdot 180^\circ = x$. Then a regular polygon with $n + 2$ sides has interior angles that measure $\frac{n+2-2}{n+2} \cdot 180 = \frac{n}{n+2} \cdot 180^\circ$, which equals $x + 6 = \frac{n-2}{n} \cdot 180 + 6^\circ$. Set these two expressions equal and solve for n :

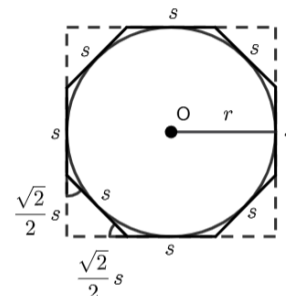
$$\begin{aligned} \frac{n}{n+2} \cdot 180 &= \frac{n-2}{n} \cdot 180 + 6 \\ 180 \left(\frac{n}{n+2} - \frac{n-2}{n} \right) &= 6 \\ 30 \left(\frac{n^2}{n(n+2)} - \frac{(n+2)(n-2)}{n(n+2)} \right) &= 1 \\ 30 \frac{n^2 - (n^2 - 4)}{n^2 + 2n} &= 1 \\ 30 \frac{4}{n^2 + 2n} &= 1 \\ 30 \cdot 4 &= 120 = n^2 + 2n \end{aligned}$$

and $n^2 + 2n - 120 = 0$. Factor this equation: $(n + 12)(n - 10) = 0$ and $n = 10$ because the extraneous solution $n = -12$ is impossible. A 10 sided regular polygon has interior angles that measure $\frac{10-2}{10} \cdot 180 = \frac{8}{10} \cdot 180 = 8 \cdot 18 = 144^\circ = x$, and $(n, x) = \boxed{(10, 144)}$.

Alternative solution: Recall that a regular polygon with n sides has exterior angles with measure $\frac{360^\circ}{n}$. Since the sum of an interior and an exterior angle equals 180° for any polygon, the polygon with $n + 2$ sides will have exterior angles that measure 6° less than the exterior angles of the polygon with n sides. Then $\frac{360}{n} = \frac{360}{n+2} + 6$, $360 \left(\frac{1}{n} - \frac{1}{n+2} \right) = 6$, $60 \left(\frac{n+2-n}{n(n+2)} \right) = 60 \left(\frac{2}{n^2+2n} \right) = 1$ and $n^2 + 2n = 120$ as in the first solution, leading to the same answer.

3. A circle is inscribed in a regular octagon. The ratio of the area of the octagon to the area of the circle can be expressed as $\frac{x}{\pi}$. Find x .

Solution: Draw circle O inscribed in a regular octagon. Let s be the length of a side of the octagon and let r be the radius of the circle. Extend every other side until it intersects in each direction with another line containing a side, shown as dotted lines in the figure at right. Note that each extended line forms an exterior angle with each of the two adjacent sides. Recall that the exterior angles of an n sided polygon have a measure of $\frac{360^\circ}{n}$. In this case, $n = 8$ and these angles measure $\frac{360}{8} = 45^\circ$. Observe that each of the four triangles formed by extended sides and an adjacent side forms a 45-45-90 (isosceles right) triangle. Recall that the sides of a 45-45-90 triangle are in proportion $1 : 1 : \sqrt{2}$. Multiply this proportion by $\frac{\sqrt{2}}{2}s$ and the side lengths are $\frac{\sqrt{2}}{2}s$, $\frac{\sqrt{2}}{2}s$, and s , as marked on the figure.



Observe that the polygon with four dotted line sides is a square, and that the area of the octagon is equal to the area of the square minus the areas of the four right triangles, which sum to:

$$4 \cdot \frac{1}{2} \cdot \left(\frac{\sqrt{2}}{2}s \right)^2 = \frac{4(\sqrt{2})^2}{2(2)^2} s^2 = 2 \cdot \frac{2}{4} = \frac{4}{4} s^2 = s^2$$

The side length of the (dotted line) square equals $\frac{\sqrt{2}}{2}s + s + \frac{\sqrt{2}}{2}s = 2 \cdot \frac{\sqrt{2}}{2}s + s = s\sqrt{2} + s = s(\sqrt{2} + 1)$. Therefore the area of the square equals

$$\begin{aligned} \left(s(\sqrt{2} + 1) \right)^2 &= s^2 (\sqrt{2} + 1)^2 = s^2 ((\sqrt{2})^2 + 2\sqrt{2} + 1) \\ &= (2 + 2\sqrt{2} + 1) s^2 = (3 + 2\sqrt{2}) s^2 \end{aligned}$$

and the area of the octagon equals $(3 + 2\sqrt{2}) s^2 - s^2 = (3 - 1 + 2\sqrt{2}) s^2 = (2 + 2\sqrt{2}) s^2$.

Next, note that the radius of the circle r is equal to half the side length of the square, or $r = \frac{s(\sqrt{2}+1)}{2}$.

Then the area of the circle equals $\pi r^2 = \pi \frac{s^2(\sqrt{2}+1)^2}{2^2} = \pi s^2 \frac{3+2\sqrt{2}}{4}$. The ratio of the octagon area to the circle area is therefore:

$$\begin{aligned} \frac{(2 + 2\sqrt{2}) s^2}{\pi s^2 \frac{3+2\sqrt{2}}{4}} &= \frac{4(2 + 2\sqrt{2})}{\pi(3 + 2\sqrt{2})} = \frac{4(2 + 2\sqrt{2})(3 - 2\sqrt{2})}{\pi(3 + 2\sqrt{2})(3 - 2\sqrt{2})} \\ &= \frac{4(6 + 6\sqrt{2} - 4\sqrt{2} - (2\sqrt{2})^2)}{\pi(3^2 - (2\sqrt{2})^2)} = \frac{4((6 - 4)\sqrt{2} + 6 - 8)}{\pi(9 - 8)} \\ &= \frac{4(2\sqrt{2} - 2)}{\pi} = \boxed{\frac{8\sqrt{2} - 8}{\pi}} \end{aligned}$$

Round 4 - Sequences and Series

1. The numbers 2, n , and 18, in that order, form a geometric sequence. Find the smallest possible value of n .

Solution: Let r be the common ratio of the geometric sequence; that is, every number but the first number in the sequence is equal to r times the number immediately preceding it. Then $n = r \cdot 2$ and $18 = r \cdot n = r \cdot (r \cdot 2) = r^2 \cdot 2$, $r^2 = 18 \div 2 = 9$, and $r = \pm\sqrt{9} = \pm 3$. Finally, $n = 2r = 2(\pm 3) = \pm 2 \cdot 3 = \pm 6$. The smallest of 6 and -6 is clearly $\boxed{-6}$.

2. Find S , the sum of the arithmetic sequence shown below:

$$S = 13 + 17 + 21 + \cdots + 217 + 221$$

Solution: Recall that the arithmetic mean of a finite arithmetic sequence $x_1, x_2, x_3, \dots, x_n$ can be found by taking the average of any two terms in opposing positions in the sequence: x_1 and x_n for instance, or x_k and x_{n-k} for any $k < n$. Recall also that the sum S_n of the first n terms of that sequence is equal to n times the arithmetic mean of the sequence, or $S_n = \frac{n(x_1+x_n)}{2}$.

To calculate S , it remains to find the number of terms in the sequence (n). Recall that $x_n = x_1 + (n-1)d$ for an arithmetic sequence, where d is the common difference, that is, $d = x_k - x_{k-1}$. Here $d = x_2 - x_1 = 17 - 13 = 4$, $x_n = 221$ and $x_1 = 13$. Replace these values in the x_n equation:

$$\begin{aligned} 221 &= 13 + 4(n-1) \\ 221 - 13 &= 208 = 4(n-1) \\ \frac{208}{4} &= 52 = n-1 \end{aligned}$$

and $n = 52 + 1 = 53$. Then $S = \frac{n(x_1+x_n)}{2} = \frac{53(13+221)}{2} = \frac{53 \cdot 234}{2} = 53 \cdot 117 = (50+3)(120-3) = 50 \cdot 120 + 3 \cdot 120 - 3 \cdot 50 - 3 \cdot 3 = 6000 + 360 - 150 - 9 = 6210 - 9 = \boxed{6201}$.

3. Let $\sqrt{1+2+3+\cdots+n}$ be a positive integer (as is n). Find the sum of the two smallest possible values for n , where $n > 1$.

Solution: Let $k = \sqrt{1+2+3+\cdots+n}$ so that $k^2 = 1+2+3+\cdots+n$. Note that if $n > 1$ then $k > 1$. Recall that the sum of the first n integers is equal to $S_n = \frac{n(n+1)}{2}$, a special case of the formula for the sum of the first n terms in an arithmetic sequence. Then $k^2 = S_n = \frac{n(n+1)}{2}$ and $n(n+1) = 2k^2$.

Proceed by testing for increasing values of k , calculating $2k^2$ and checking to see if it can be factored into two consecutive integers. Note that k cannot be prime, because then $2k^2 = k \cdot 2k$ cannot be factored into successive integers. Nor can it be a perfect square (other than $k = 1$) for the same reason. Finally, $2k^2$ must be factored into an odd and an even number, so k must have an odd factor.

Test composite numbers k that are not prime, not perfect squares, and have at least one odd factor, looking for an odd/even factor pair. Note that composite numbers of the form $k = 2p$, where p is prime and $p > 3$, will fail the test: factoring $2k^2 = 2(2p)^2$ into an odd/even pair results in $p^2 \cdot 8$ or $p \cdot 8p$, neither of which are consecutive integers. Therefore 10, 14, 22, 26, and 34 need not be tested. The results from testing the first few candidates are:

k	k^2	$2k^2$	factor	$n(n+1)$?
6	36	72	$8 \cdot 9$	yes
12	144	288	$9 \cdot 32$	no
15	225	450	$15 \cdot 30$	no
18	324	628	$24 \cdot 27$	no
20	400	800	$25 \cdot 32$	no
21	441	882	$21 \cdot 42$	no
24	576	1152	$24 \cdot 48$	no
27	729	1458	$27 \cdot 54$	no
28	784	1568	$32 \cdot 49$	no
30	900	1800	$40 \cdot 45$	no
33	1089	2178	$33 \cdot 66$	no
35	1225	2450	$49 \cdot 50$	yes

Therefore the two smallest values of n , $n > 1$, are 8 and 49. Their sum is $8 + 49 = \boxed{57}$.

Round 5 - Matrices and Systems of Equations

1. Solve for k :

$$\begin{vmatrix} -3 & k \\ 7 & -8 \end{vmatrix} = 10$$

Solution: Recall that the determinant of a two by two matrix is equal to the product of the elements on its diagonal minus the product of the elements on its cross diagonal. Specifically:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Therefore

$$\begin{vmatrix} -3 & k \\ 7 & -8 \end{vmatrix} = (-3)(-8) - k \cdot 7 = 24 - 7k = 10.$$

Add $7k - 10$ to both sides of the last equation: $24 - 10 = 14 = 7k$. Then $k = 14 \div 7 = \boxed{2}$.

2. Given the system of three equations $\frac{1}{x} + \frac{1}{y} = \frac{3}{2}$, $\frac{1}{x} + \frac{1}{z} = \frac{4}{3}$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{2}$: find the ordered triple (x, y, z) :

Solution: Let $u = \frac{1}{x}$, $v = \frac{1}{y}$, and $w = \frac{1}{z}$, substitute these for x , y , and z in the initial system of equations, and multiply the resulting equations by 2, 3, and 2, respectively, to eliminate fractions:

$$2u + 2v = 3$$

$$3u + 3w = 4$$

$$2v + 2w = 1$$

Subtract the third equation from the first equation: $2u + 2v - 2v - 2w = 3 - 1$, or $2u - 2w = 2$. Multiply this equation by $\frac{3}{2}$: $\frac{3}{2}(2u - 2w = 2)$ so that $3u - 3w = 3$, leaving two equations in unknowns u and w :

$$3u + 3w = 4$$

$$3u - 3w = 3$$

Add these two equations to eliminate w : $3u + 3u + 3w - 3w = 6u = 4 + 3 = 7$. Therefore $u = \frac{7}{6}$. From the second equation above, $3w = 3u - 3$, or $w = u - 1$, so $w = \frac{7}{6} - 1 = \frac{1}{6}$. Next, rearrange the first of the three equations and $2v = 3 - 2u = 3 - 2\left(\frac{7}{6}\right) = \frac{18-14}{6} = \frac{4}{6} = \frac{2}{3}$, so $v = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

Finally, take the reciprocals of x, y , and z : $x = \frac{1}{u} = \frac{6}{7}$, $y = \frac{1}{v} = 3$, and $z = \frac{1}{w} = 6$. Then

$$(x, y, z) = \left(\frac{6}{7}, 3, 6\right).$$

3. Let

$$\begin{bmatrix} 2 & b \\ a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for some x and y , where a and b are integers and at least one of x and y does not equal zero. Find the sum of $a + b$ values over all possible ordered pairs (a, b) .

Solution: This matrix equation describes a system of two equations in two unknowns. Multiply each row of the 2 X 2 matrices by the (x, y) column vector to write two equations:

$$2x + by = 6x - y$$

$$ax + 3y = 4x + 6y$$

Collect terms on the right hand side of each equation by subtracting the left hand side expressions:

$$0 = (6 - 2)x + (-b - 1)y = 4x - (b + 1)y$$

$$0 = (4 - a)x + (6 - 3)y = (4 - a)x + 3y$$

Note that $x = 0$ and $y = 0$, the *trivial* solution, solves this system for any (a, b) pair. Subtract $\frac{4-a}{4}$ times the top right expression from the bottom right expression: $\left(3 + \frac{(4-a)(b+1)}{4}\right)y = 0$. Multiply this equation by 4: $(12 + (4 - a)(b + 1))y = 0$. If the expression in parentheses does *not* equal zero, then $y = 0$. Substitute $y = 0$ into the first equation, and $x = 0$. Therefore if a nontrivial solution exists, $12 + (4 - a)(b + 1)$ *must* equal 0, or

$$(4 - a)(b + 1) = -12.$$

Recall that both a and b are integers and observe that $(4 - a)$ and $(b + 1)$ must be integers whose product is -12 . There are a finite numbers of such pairs. The possible values of $4 - a$ are

$$(4 - a) \in \{-1, -2, -3, -4, -6, -12, 1, 2, 3, 4, 6, 12\},$$

and pairing these the same values in reverse order will give all ordered pairs of integers $(4 - a, b + 1)$ that meet the condition. There are twelve $(4 - a, b + 1)$ pairs that correspond to twelve (a, b) pairs. Next, observe that both the sum of all $(4 - a)$ values and the sum of all $(b + 1)$ values equal 0. Let A be the sum of the a values and B be the sum of the b values, so that the sum of all (a, b) pairs equals $A + B$. Then the sum of the $(4 - a)$ values is $12 \cdot 4 - A = 0$ and $A = 48$. Likewise the sum of the $(b + 1)$ values is $B + 12 \cdot 1 = 0$ and $B = -12$. Finally, $A + B = 48 - 12 = \boxed{36}$.

Alternative solution: Recall that a system of n *homogeneous linear* equations ($A\underline{x} = \underline{0}$ where $\underline{x}$ is the vector of unknowns and $\underline{0}$ is the *zero* vector) in n unknowns will have a *nontrivial* solution only if the determinant of the *system matrix* (A) is zero. Subtract the left hand side of the original matrix equation from the right hand side to put the equation in form $A\underline{x} = \underline{0}$:

$$\begin{bmatrix} 6 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 & b \\ a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\begin{bmatrix} 6 & -1 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & b \\ a & 3 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -1-b \\ 4-a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find the determinant of the system matrix by subtracting the product of the cross diagonal elements from the product of the main diagonal elements and $4 \cdot 3 - (-1 - b)(4 - a) = 12 + (4 - a)(b + 1) = 0$ as in the above solution.

Team Round

1. Let $x = \left(\frac{3}{4} + \frac{1}{6}\right) \left(\frac{3}{4} - \frac{1}{6}\right)$. Express x as a fraction in simplest terms.

Solution: Start by putting the fractions over a common denominator:

$$\left(\frac{3}{4} + \frac{1}{6}\right) \left(\frac{3}{4} - \frac{1}{6}\right) = \left(\frac{3 \cdot 3}{4 \cdot 3} + \frac{1 \cdot 2}{6 \cdot 2}\right) \left(\frac{3 \cdot 3}{4 \cdot 3} - \frac{1 \cdot 2}{6 \cdot 2}\right) = \left(\frac{9}{12} + \frac{2}{12}\right) \left(\frac{9}{12} - \frac{2}{12}\right).$$

Next, follow order of operations and $\frac{11}{12} \cdot \frac{7}{12} = \boxed{\frac{77}{144}}$.

Alternative solution: Recall and apply the difference of squares identity $((a+b)(a-b) = a^2 - b^2)$:

$$\left(\frac{3}{4} + \frac{1}{6}\right) \left(\frac{3}{4} - \frac{1}{6}\right) = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{6}\right)^2 = \frac{9}{16} - \frac{1}{36}.$$

Next, find the common denominator ($16 \cdot 9 = 144$) and subtract: $\frac{9 \cdot 9}{16 \cdot 9} - \frac{1 \cdot 4}{36 \cdot 4} = \frac{81}{144} - \frac{4}{144} = \frac{77}{144}$.

2. Let $ab = a - b$ where a and b are two non-zero real numbers. Find

$$\frac{a}{b} + \frac{b}{a} - ab$$

Solution: To begin, square both sides of the given identity ($ab = a - b$):

$$(ab)^2 = a^2b^2 = (a - b)^2 = a^2 - 2ab + b^2.$$

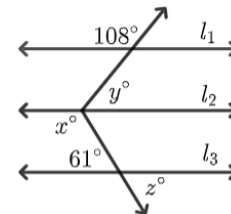
Next, divide both sides of $a^2b^2 = a^2 - 2ab + b^2$ by ab (possible because $ab \neq 0$):

$$\frac{a^2b^2}{ab} = ab = \frac{a^2 - 2ab + b^2}{ab} = \frac{a^2}{ab} - \frac{2ab}{ab} + \frac{b^2}{ab} = \frac{a}{b} - 2 + \frac{b}{a}$$

Lastly, add $2 - ab$ to both sides of the equation $\frac{a}{b} - 2 + \frac{b}{a} = ab$ and:

$$\frac{a}{b} + \frac{b}{a} - ab = \boxed{2}$$

3. Given three parallel lines in the figure at right, that is $l_1 \parallel l_2 \parallel l_3$, and the angle measures marked in the figure, find the value of $\frac{x-y}{z}$



Solution: Note that the angles marked with z° and 61° are vertical angles, so $z = 61^\circ$ because vertical angles have equal measures. Second, note that the angles marked with x° and 61° are alternate interior angles. Recall that alternate interior angles are supplementary (their measures sum to 180°) so that $x + 61 = 180^\circ$ and $x = 180 - 61 = 119^\circ$. Third, the measures of the angle marked 108° and its supplement located beneath l_1 sum to 180° . Then the supplement measures $180 - 108 = 72^\circ$. This supplement and the angle marked y° are alternate interior angles, which have equal measures. Therefore $y = 72^\circ$.

Lastly, substitute the three values in the desired expression: $\frac{x-y}{z} = \frac{119-72}{61} = \boxed{\frac{47}{61}}$.

4. Find integer n where:

$$\frac{1 + 2 + 3 + \cdots + n}{3n} = 36.$$

Solution: The sum of the first n integers, $1 + 2 + 3 + \cdots + n$ is equal to $\frac{n(n+1)}{2}$: the number of terms n times average term, the sum of the first term and last terms divided by 2. Replace the sum with this expression, simplify, and solve for n :

$$\frac{n(n+1)}{2 \cdot 3n} = \frac{n+1}{6} = 36$$

$$n+1 = 6 \cdot 36 = 216$$

and $n = 216 - 1 = \boxed{215}$

5. Express x as a decimal if:

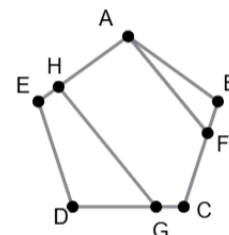
$$x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + 1}}}$$

Solution: Evaluate the expression starting with $1 + 1 = 2$, add 1 to its reciprocal, and repeat this process three times. Note that $1 + \frac{a}{b} = \frac{a+b}{b}$. Then $1 + \frac{1}{1+1} = 1 + \frac{1}{2} = \frac{1+2}{2} = \frac{3}{2}$. Next, $1 + \frac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3}$. Finally, $x = 1 + \frac{1}{\frac{5}{3}} = 1 + \frac{3}{5} = \frac{8}{5} = \boxed{1.6}$, expressed as a decimal.

6. How many ml of water must be added to $60ml$ of a 12% alcohol solution to produce a 9% solution?

Solution: The $60ml$ of 12% alcohol solution contains $60ml \cdot 0.12 = 7.2ml$ of alcohol. For the $7.2ml$ of alcohol to be 9% of the total solution volume V , $0.09V$ must equal $7.2ml$ of alcohol. Thus, $V \cdot 0.09 = 7.2ml$. Divide both sides by 0.09 and $V = 7.2ml \div 0.09 = .8ml \div 0.01 = .8ml \cdot 100 = 80ml$. Therefore $80ml - 60ml = \boxed{20}ml$ of water must be added.

7. $ABCDE$, shown at right, is a regular pentagon. F lies on $\overline{BC}$, G lies on $\overline{CD}$, and H lies on $\overline{EA}$. Also, $\overline{AF} \parallel \overline{HG}$ and $m\angle BAF = 22^\circ$. Find $m\angle HGD$ in $^\circ$.



Solution: First, recall that the sum of the interior angles of a n -sided polygon equals $(n - 2) \cdot 180^\circ$. For a pentagon, this sum equals $3 \cdot 180^\circ = 540^\circ$. Therefore each angle of a regular pentagon measures $\frac{540^\circ}{5} = 108^\circ$. Observe that adjacent angles $\angle FAH$ and $\angle BAF$ form an interior angle of $ABCDE$, so that $m\angle FAH + m\angle BAF = 108^\circ$. Then $m\angle FAH = 108^\circ - m\angle BAF = 108^\circ - 22^\circ = 86^\circ$ after substituting the given value for $m\angle BAF$. Since $\overline{AF} \parallel \overline{HG}$ and $\angle FAH$ and $\angle GHE$ are corresponding angles, $m\angle GHE = m\angle BAF = 86^\circ$.

Next, observe that the sum of interior angles of quadrilateral $HEDG$ equals $2 \cdot 180^\circ = 360^\circ$ because $n = 4$. Also observe that $m\angle HEG = m\angle EDG = 108^\circ$ because they are interior angles of $ABCDE$. Then $360^\circ = m\angle GHE + m\angle HEG + m\angle EDG + m\angle HGD = 86^\circ + 108^\circ + 108^\circ + m\angle HGD$. Thus, $m\angle HGD = 360^\circ - 86^\circ - 108^\circ - 108^\circ = \boxed{58^\circ}$

8. Find $S = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$. Express your answer as a fraction in simplest form.

Solution: Start by using the method of *partial fraction expansion* to break the summation argument into two fractions:

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}$$

where $A(n-1) + B(n+1) = (A+B)n - A + B = 1$. Then $A+B=0$ and $B-A=1$. Add these two equations: $2B=1$, and $B=\frac{1}{2}$. Substitute B into either of the two equations and $A=-\frac{1}{2}$. Therefore $\frac{1}{n^2-1} = -\frac{1}{2(n+1)} + \frac{1}{2(n-1)}$, which can be readily verified.

Substitute this expression for the summation argument, apply the identity $\sum_n (x_n + y_n) = \sum_n x_n + \sum_n y_n$ to break the original summation into two summations, and continue by factoring out $\frac{1}{2}$:

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \sum_{n=2}^{\infty} \frac{-1}{2(n+1)} + \sum_{n=2}^{\infty} \frac{1}{2(n-1)} \\ &= \sum_{n=2}^{\infty} -\frac{1}{2} \cdot \frac{1}{n+1} + \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{1}{n-1} \\ &= -\frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n+1} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n-1} \\ &= \frac{1}{2} \left(\sum_{n=2}^{\infty} \frac{1}{n-1} - \sum_{n=2}^{\infty} \frac{1}{n+1} \right) \end{aligned}$$

Let S_l be the left summation and S_r be the right summation inside the parentheses of the above expression. Then $S_l = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = 1 + \frac{1}{2} + S_r$. The difference of the two summations is $S_l - S_r = 1 + \frac{1}{2} + S_r - S_r = \frac{3}{2}$. Finally, $S = \frac{1}{2} (S_l - S_r) = \frac{1}{2} \cdot \frac{3}{2} = \boxed{\frac{3}{4}}$.

9. Find matrix A where

$$A \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} + \left(\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} \right)^T = \begin{bmatrix} 1 & 8 \\ 2 & 2 \end{bmatrix}$$

Solution: First, recall that the inverse of 2 X 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, which is easily verified by multiplying the original matrix by its inverse and checking to see if the result is the identity matrix. Apply this formula to the second matrix, and then take the transpose of the result:

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} \right)^T = \left(\frac{1}{1 \cdot 5 - 2 \cdot 3} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} \right)^T = \frac{1}{-1} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = - \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$

Next, add -1 times this matrix to both sides of the original equation, removing it from the left hand side:

$$A \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix}$$

Next, find the inverse of the matrix that multiplies A on the right:

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{3 \cdot 1 - 0 \cdot 2} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

Note that the product of a matrix and its inverse is the identity matrix I , and also that $AI = A$.

Therefore multiply both sides of the preceding equation *on the right side* by $\frac{1}{3} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ to find A:

$$\begin{aligned} A \left(\frac{1}{3} \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \right) &= \frac{1}{3} \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \\ A \left(\frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) &= \frac{1}{3} \begin{bmatrix} 6 & 3 \\ 0 & 9 \end{bmatrix} \\ A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= A = \boxed{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}. \end{aligned}$$

Note that order doesn't matter when a matrix is multiplied by a scalar. In the above equations, the scalar was placed for clarity without affecting the correctness of the solution.